MATH 245 S17, Exam 1 Solutions

1. Carefully define the following terms: composite, conjunction, tautology, Double Negation semantic theorem

Let $n \in \mathbb{Z}$ with $n \geq 2$. We call n composite if there is some $a \in \mathbb{Z}$ such that $1 < a < n$ and $a|n$. Let p, q be propositions. Their conjunction is the proposition that is true if p, q are both true, and false otherwise. A tautology is a (compound) proposition that is always true. The Double Negation semantic theorem states that for every proposition p, we have $\neg(\neg p) \equiv p$.

2. Carefully define the following terms: Addition semantic theorem, Trivial Proof theorem, Direct Proof, converse.

The Addition semantic theorem states that for any propositions p, q, we have $p \vdash p \vee q$. The Trivial Proof theorem states that for any propositions p, q, we have $q \vdash p \to q$. The Direct Proof theorem states that for any propositions p, q, if $p \vdash q$ is valid, then $p \to q$ is true. The converse of conditional proposition $p \to q$ is $q \to p$.

3. Calculate and simplify $\frac{(\lfloor 13.9 \rfloor + \lfloor -1.2 \rfloor)!}{\lceil 8.4 \rceil !}$.

We have $\frac{(\lfloor 13.9 \rfloor + \lfloor -1.2 \rfloor)!}{\lceil 8.4 \rceil !} = \frac{(13-2)!}{9!} = \frac{11!}{9!} = \frac{11 \cdot 10 \cdot 9!}{9!} = 11 \cdot 10 = 110$.

4. Let $a, b, c \in \mathbb{Z}$. Suppose that $a|b$ and $a|c$. Prove that $a|(b + c)$.

Because $a|b$ there is some $m \in \mathbb{Z}$ with $b = ma$. Because $a|c$ there is some $n \in \mathbb{Z}$ with $c = na$. Adding, we get $b + c = ma + na = (m + n)a$. Now $a|(b + c)$ because $m + n \in \mathbb{Z}$.

5. Use truth tables to prove the half of De Morgan's Law which states that for any propositions p, q we have $\neg(p \vee q) \equiv (\neg p) \wedge (\neg q)$.

p	q	$p \vee q$	$\neg(p \vee q)$	$\neg p$	$\neg q$	$(\neg p) \wedge (\neg q)$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

The fourth and seventh columns, as highlighted, agree. Hence $\neg(p \vee q) \equiv (\neg p) \wedge (\neg q)$.

6. Simplify $\neg((p \to q) \wedge r)$ as much as possible. (i.e. where only basic propositions are negated)

We start with $\neg((p \to q) \wedge r)$. Applying conditional interpretation, this is equivalent to $\neg((q \vee \neg p) \wedge r)$. Applying De Morgan's Law, this is equivalent to $(\neg(q \vee \neg p)) \vee (\neg r)$. Applying De Morgan's Law again, this is equivalent to $((\neg q) \wedge \neg(\neg p)) \vee (\neg r)$. Finally, applying double negation, this is equivalent to $((\neg q) \wedge p) \vee (\neg r)$.

7. Let $x \in \mathbb{R}$. Prove that if x is irrational then $\frac{x}{3}$ is irrational.

We use a contrapositive proof. Assume that $\frac{x}{3}$ is rational. Then there are integers m, n, with $n \neq 0$, such that $\frac{x}{3} = \frac{m}{n}$. Multiplying both sides by 3 we get $x = \frac{3m}{n}$. Now, $3m, n$ are integers with $n \neq 0$, so x is rational.

8. Let $n \in \mathbb{Z}$. Suppose that n is even. Prove that $3n^2 + 1$ is odd.

We use a direct proof. Suppose that n is even. Then there is an integer m with $n = 2m$. Now, $3n^2 + 1 = 3(2m)^2 + 1 = 3(4m^2) + 1 = 2(6m^2) + 1$. Because $6m^2$ is an integer, $3n^2 + 1$ is odd.

9. Using semantic theorems, prove that for any propositions p, q, r, we have $((p \vee q) \vee r), (\neg q) \vdash p \vee r$.

Start with hypothesis $(p \vee q) \vee r$. Applying commutativity of $\vee$, we get $(q \vee p) \vee r$. Applying associativity of $\vee$, we get $q \vee (p \vee r)$. Now apply disjunctive syllogism to this and to hypothesis $\neg q$ to get $p \vee r$.

10. Using semantic theorems, prove that for any propositions p, q, r, we have $(p \to q), (q \to r) \vdash (p \to r)$.

If q is true, then applying modus ponens to hypothesis $q \to r$ gives r. Applying addition gives $r \vee \neg p$.

If instead q is false, then applying modus tollens to hypothesis $p \to q$ gives $\neg p$. Applying addition gives $r \vee \neg p$.

Either way we have $r \vee \neg p$. Applying conditional interpretation we get $p \to r$.